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LETTER TO THE EDITOR

Dynamics of field-driven interfaces in the two-dimensional Ising model

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Abstract. The time-dependent properties of an inclined interface separating up and down spin regions in a two-dimensional nearest-neighbour Ising model evolving under Glauber dynamics in a non-zero field are studied. In the limit of large exchange coupling, the model reduces to the single-step model for ballistic growth and thence to the asymmetric exclusion process which describes a driven diffusive system of hard core particles on a one-dimensional lattice. The drift velocity of the interface is found as a function of field, temperature and inclination, and interface correlation functions are related to sliding tag correlation functions in the particle system. The existence of a critical value of the sliding-tag velocity implies that there is an inclination-dependent easy direction along which temporal interface fluctuations grow subdiffusively. This direction is found, as is the asymptotic behaviour of the correlation function in all other directions.

There is currently a great deal of interest in the dynamical properties of moving interfaces, such as the profile resulting from deposition of material on a surface, or the boundary between two phases not in equilibrium. Monte Carlo simulations of simple lattice and off-lattice systems and studies of stochastic field theoretical models show that driven interfaces have growth laws for fluctuations that differ from their non-moving counterparts [1].

These features were illustrated in recent Monte Carlo work on the two-dimensional Ising model [2]. The object of study was the interface between the up and down spin phases at low temperature T, in the presence of an external field h. In the specific model studied, the exchange coupling was assumed to be anisotropic ($J_x \neq J_y$), and h, T and J_y were all taken to be much smaller than J_x. Under these circumstances, there are practically no overhangs, and the interface is represented, at any instant, by a directed walk oriented along the y-axis. The drift and broadening of an initially flat interface were studied numerically [2], and found to exhibit behaviour quite different from the case $h = 0$ [3, 4].

In this paper we point out that it is possible to make some analytical statements about the time evolution of an inclined interface between the low-temperature phases of an Ising model in an external field. We assume that h and $T \equiv 1/\beta$ are both much smaller than the nearest neighbour exchange coupling J (assumed isotropic).

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As in [2], there are no overhangs; here the interface is a directed walk making a specified angle ϕ with the x-axis (figure 1). However, unlike the situation considered in [2], the length of the interface here does not vary in time. In steady state, the interface sweeps across the lattice with a velocity V which depends on βh and on the inclination ϕ. With Glauber dynamics for the Ising spins, the model reduces to the single-step model of ballistic growth [5–7], and thence to the one-dimensional exclusion process—a stochastic model of moving hard-core particles on a lattice [8, 9]. In particular, dynamical interface correlation functions are related to sliding-tag correlation functions [10, 11] in the exclusion process. We use this correspondence to deduce several steady state properties of the moving interface: (a) the dependence of V on ϕ and βh; (b) the orientation θ^* (a function of ϕ) of an 'easy' direction along which fluctuations follow a slower growth law than along any other direction; (c) the asymptotic growth of fluctuations in all other directions, $\theta \neq \theta^*$.

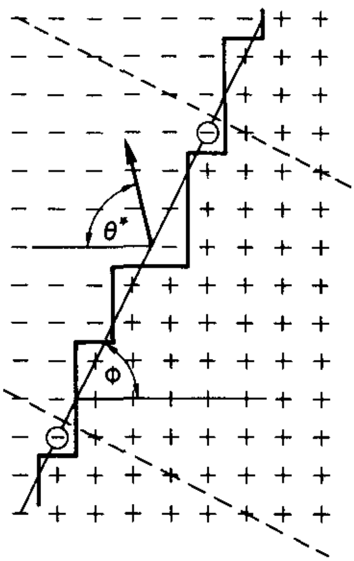

Figure 1. A tilted interface (bold line) between regions of up and down Ising spins. Cylindrical boundary conditions for the spins are defined by identifying the broken lines in the figure, so that, for instance, spins at the circled sites are equivalent. Since J is very large, only spins at the corners, like the circled ones, can flip. The arrow depicts the easy direction along which fluctuations are subdiffusive.

Since we are interested in interfaces which have a specified inclination, it is convenient to use cylindrical boundary conditions for the spins, with the axis of the cylinder making an angle $-(\pi/2 - \phi)$ with the x-axis (figure 1). The cylinder is supposed to be infinitely long, and the system is started at $t = -\infty$ in a state with a single interface separating a region of all up spins on one side of the cylinder, from a region of down spins. The field h, assumed positive for definiteness, induces the interface to move into the down-spin region. The boundary conditions ensure that on average the interface makes an angle ϕ with the x-axis . We are interested in characterizing the steady state properties of the interface in the course of its motion, in the thermodynamic limit when the radius of the cylinder is taken to infinity.

As the Ising spins obey single-spin flip Glauber dynamics, and h, $T \ll J$ holds, only spins at the corners of the interface (e.g. the circled spins in figure 1) have an appreciable probability of reversing. Such a reversal leads to a flip of the corner, which preserves the length of the interface; in the limit h/J, $T/J \to 0$, the model reduces to the single-step deposition–evaporation model, with corner flips corresponding to the elementary processes of deposition and evaporation [5, 6], or alternatively to a cellular automaton rule for the redistribution of local curvatures [7]. The mapping to the exclusion process follows if we associate a particle with each vertical bond of the interface, and a hole with each horizontal bond. Corner flip dynamics then corresponds to particle–hole exchange (Kawasaki dynamics) for the particle system. In the exclusion process, each particle is sampled once in each time step, and attempts to hop rightward with probability p, and leftward with probability q, with $p + q = 1$. Because of the hard core constraint, the hop actually takes place only if the sought site is vacant. The probabilities p and q can be related to the Ising model parameters on noting that the ratio of the rates of up–down and down–up flips is $\exp(-2\beta h)$; thus $p = \exp(\beta h)/2\cosh(\beta h)$.

A similar mapping from Ising interfaces to the exclusion process has also been discussed in the context of cluster kinetics [12, 13] but the boundary conditions employed and the questions addressed were different. Here, periodic boundary conditions allow us to think of the particle system carrying a current on a ring. Let N_{p} and N_{s} denote the number of particles and sites respectively, and let $\rho = N_{\mathrm{p}}/N_{\mathrm{s}}$ be the density. In steady state, every configuration of the N_{p} particles is equally likely [9]. Let us label the particles sequentially, and let $y_n(t)$ be the displacement of the nth particle at time t. The drift velocity in steady state is [9]

$$v_{\mathrm{p}} \equiv \langle (y_n(t) - y_n(0)) \rangle = (1-\rho)\,(p-q). \tag{1}$$

The drift velocity for labelled holes is $v_{\mathrm{h}} = -(p-q)\rho$, and the current is $j = \rho(1-\rho)(p-q)$.

The average inclination of the corresponding interface at any instant is given by $\tan\phi = \rho/(1-\rho)$ with $0 < \phi < \pi/2$. In time t, the mean vertical shift of the interface is given by the average number of particles which pass by a given hole, namely, $\rho(v_{\mathrm{p}} - v_{\mathrm{h}})t$. The magnitude of the normal velocity is then $V = \rho(v_{\mathrm{p}} - v_{\mathrm{h}})\cos\phi = j(\sin\phi + \cos\phi)$ which may be rewritten as

$$V = \frac{\tanh\beta h}{\sec\phi + \operatorname{cosec}\phi}. \tag{2}$$

The correlation function involving interface fluctuations is extremely anisotropic, and the easy direction along which fluctuations are smallest, is not, in general, normal to the interface. Noting that the abscissa and ordinate of a point on the interface are just hole and particle tag labels respectively, it is natural to consider the effect of sliding tags [10, 11], i.e. to consider correlation functions which monitor fluctuations in the separation between a tagged particle (at $t = 0$) and a particle whose tag increases linearly in time

$$\sigma^2(u,t) = \langle (y(n_t,t) - y(n,0) - ut)^2 \rangle \tag{3}$$

with $n_t = n + \rho(u - v_{\mathrm{p}})t$. Here u is a variable velocity, whose value controls the rate of tag sliding. If $u = v_{\mathrm{p}}$, then σ reduces to the auto-tag correlation function σ_{self}, for which the following exact results for the $t \to \infty$ behaviour are known [9, 14]

$$\sigma^2_{\mathrm{self}} = (1-\rho)\,(p-q)\,t \qquad (p \neq q) \tag{4a}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{(1-\rho)}{\rho}\, t^{1/2} \qquad (p = q = 1/2). \tag{4b}$$

For general u, the sliding tag correlation function (3) has been studied in [10]. A coarse-grained description based on the Kardar–Parisi–Zhang (KPZ) equation [15] with a first order gradient term included suggests that there is a critical value u_c at which fluctuations are suppressed [10]. Choosing $u = u_c$ corresponds to eliminating the first order gradient term which is present if $u \neq u_c$. The resulting asymptotic behaviour of the correlation function is

$$\sigma^2(u,t) \sim t \qquad (u \neq u_c) \tag{5a}$$

$$\sigma^2(u_c,t) \sim t^{2/3}. \tag{5b}$$

The value $\frac{2}{3}$ of the exponent characterizing the subdiffusive growth in (5*b*) was deduced from a renormalization group study of the noisy Burgers equation [15, 16], and confirmed by a Bethe ansatz calculation of the spectral gap of the stochastic evolution operator for the asymmetric exclusion process with $p = 1$ [17, 18].

A constraint on the value of u_c comes from particle–hole duality, which implies $u_c(\rho) = -u_c(1-\rho)$. The explicit answer, based on the argument given below, and originally deduced from a Monte Carlo study [10], is

$$u_c(\rho) = (1-2\rho)\,(p-q). \tag{6}$$

The velocity u_c has a simple physical interpretation. The exclusion process is known to support hydrodynamic waves which involve moving density fluctuations [19, 20], akin to longitudinal sound waves in an equilibrium system. The existence of these waves—which are examples of the 'kinematic' waves introduced and discussed in [21]—follows from the conservation of particle number in the exclusion process. For such a wave a general argument determines the velocity to be $\partial j/\partial\rho$ [20, 21]. A front (shock) which separates regions characterized by (j_1, ρ_1) and (j_2, ρ_2) has a velocity U which must satisfy $j_1 - \rho_1 U = j_2 - \rho_2 U$ since all particles of the first type which cross the front become particles of the second type. Considering adjoining regions which differ only infinitesimally in their densities, we see that the front between them has a speed $U = \partial j/\partial\rho$. In the case at hand, this equals $u_c(\rho)$ given by (6).

The existence of a well defined velocity for fronts implies that density fluctuations are transported through the system more or less intact at the same velocity. In the steady state of the exclusion process there are statistical inhomogeneities of density throughout the system, characteristic of product measure states with random occupation of sites. Consequently, in our system with periodic boundary conditions, the density pattern corresponding to the initial state is moved bodily through the system with speed u_c. We can now understand why the sliding tag correlation function shows very different behaviour depending on whether or not $u = u_c$. The choice $u = u_c$ in (3) corresponds to sliding the tags at a rate which keeps abreast of the moving density pattern. The corresponding correlation function $\sigma(u_c, t)$ then monitors the slow intrinsic decay of the density wave and the corresponding fluctuations (5*b*) are the smallest possible. If u is not equal to u_c, a phase difference develops between the sliding tags and the moving density pattern. The result is a rapid growth of $\sigma(u,t)$ which reflects, predominantly, the spatial structure of the initial density pattern which is moving past the particles.

An important consequence of the slide of the initial pattern is that it permits $\sigma(u,t)$ to be computed easily. Noting that the displacement of particle m at time

t from its mean position will mirror that of particle m' at $t = 0$, with $m' = m - \rho(u_c - v_p)t$, we may write $\sigma(u,t)$ in terms of an equal-time correlation function

$$\sigma^2(u,t) = \langle (y(n^*,0) - y(n,0) - (u - u_c)t)^2 \rangle \tag{7}$$

where $n^* = n + \rho(u - u_c)t$. Since the initial state corresponds to random occupation of sites with probability ρ, the right-hand side of (7) can be evaluated. Let n_p and n_h be the number of particles and holes between sites n and n^*. Then $y(n^*,0) - y(n,0) = n_p + n_h$, and σ^2 can be written as

$$\sigma^2(u,t) = \frac{1}{\rho^2}\langle (\rho n_h - (1-\rho)n_p)^2 \rangle \tag{8}$$

subject to $\langle n_p + n_h \rangle = (u - u_c)t$. For large t, (8) can be evaluated using the central limit theorem, with the result

$$\sigma^2(u,t) = \frac{(1-\rho)}{\rho}\,|u - u_c|\,t \qquad (u \neq u_c). \tag{9}$$

The derivation of (9) neglects the decay of the density pattern, but this is justified if t is large enough, since the resulting leading order result is larger than the contribution (5*b*) from dissipation. A non-trivial check is obtained by setting $u = v_p$, in which case the auto-tag correlation function [14] σ_{self} in (4*a*) should be recovered. This is indeed the case; (9) gives the generalization of the auto-tag diffusion constant for the sliding-tag process.

We now turn to implications for the interface in the Ising model. Density waves in the exclusion process correspond to transverse excitations of the interface, which consequently have a well-defined velocity u_c. Only when the interface tilt angle $\phi = \pi/4$, is this velocity zero. Let us measure interface parameters with respect to the average orientation AA' of the interface at $t = 0$. At time t, let $h(r,t)$ be the interface location (measured normal to AA') minus Vt, at a point a distance r along AA'. Consider the correlation function

$$S^2(\theta,t) = \langle (h(r,0) - h(r',t))^2 \rangle \tag{10}$$

where θ is the angle the displacement $(r' - r)$ makes with the negative x axis. $S(\theta,t)$ may be related to the correlation function $\sigma(u,t)$ defined in (3). The corresponding sliding tag velocity u is determined by noting that the slope is given by the ratio of the u-induced particle tag shift $\rho(u - v_p)$ (vertical displacement), to the analogous hole tag shift

$$\tan\theta = -\frac{\rho}{1-\rho}\,\frac{u - v_p}{u - v_h}. \tag{11}$$

The fluctuation in the location y_n of a given particle n gives the horizontal excursion of the interface at ordinate equal to n, and is related to the interface height h by a factor $\sin\phi$. Thus we have

$$S(\theta,t) = \sin\phi\;\sigma(u,t). \tag{12}$$

Equations (11) and (12), together with the results for the sliding tag correlation function $\sigma(u,t)$, determine the asymptotic behaviour of interface fluctuations in the Ising model.

The critical velocity u_c determines the direction θ^* of the easy axis (figure 1). From (6) and (11) we find

$$\tan\theta^* = \tan^2\phi. \tag{13}$$

Along most directions, $(\theta \neq \theta^*)$, the correlation function $S(\theta,t)$ grows as $t^{1/2}$, with a prefactor known from (9) and (12). On the other hand, along the critical direction, (5*b*) implies that $S(\theta^*,t)$ follows a $t^{1/3}$ growth law. Except when $\phi = \pi/4$, the easy direction does not coincide with the normal. This is reminiscent of the difference between the angle of incidence and the angle of growth in deposition processes. A heuristic 'tangent rule' has been proposed for ballistic deposition [22], but it is not exact [5] and differs from (13) in both form and content. The existence of an easy direction for the single-step model has not been discussed earlier, probably because previous studies [5–7] dealt only with substrates which correspond to $\phi = \pi/4$. We also note that several studies on this and related models focus on the time-dependence of the average width of an initially flat interface rather than two-point correlation functions; the width is not sensitive to transverse wave motion or the concomitant anisotropy.

The mapping between the interface and exclusion process is useful for other correlation functions too. For instance, the interface velocity–velocity correlation functions [7] are determined by sliding tag correlation functions of the form $\langle (v_{n_t}(t) - v_n(0))^2\rangle$, where v_n is the instantaneous velocity of the nth particle and n_t is the sliding tag at time t. Consequently these correlation functions too would be expected to exhibit angular anisotropy characteristized by the easy direction θ^*.

Throughout, we have been primarily concerned with fluctuation properties of a non-equilibrium interface driven by a non-zero field h, but it is also of interest to see what the mapping to the particle problem implies for interface fluctuations when $h = 0$. In this case, the interface is in thermal equilibrium, and does not move bodily $(V = 0)$. In the corresponding symmetric exclusion process, there is no macroscopic current, nor are there moving density waves. The auto-tag correlation function (4*b*) follows a $t^{1/4}$ growth law. The correlation function for two fixed tags, which determines the interface correlation function $S(r - r', t)$ has also been calculated [11] in a harmonic stochastic theory, which seems to give exact results (as checked by Monte Carlo simulation). The result for S has a scaling form with argument $z \equiv t/(r - r')^2$, and the scaling function was evaluated explicitly in [11]. The associated leading behaviour of the correlation function is $S \sim |r - r'|^{1/2}$ for $z \ll 1$, and $S \sim t^{1/4}$ for large times $(z \gg 1)$. This behaviour is in accord with Monte Carlo results for the anisotropic Ising model when $h = 0$ [3], and is characteristic of fluctuations in surfaces in which growth-induced nonlinearities are not important [4, 23, 24].

In summary, the mapping from the Ising interface to the exclusion process of moving hard core particles allows an exact determination of several interfacial quantities of interest, as functions of βh and the inclination ϕ. A simple physical picture underlies the dynamics. The number of particles in the exclusion process (or correspondingly the length of the horizontal and vertical segments of the interface in the Ising problem) is conserved, and an immediate consequence of the conservation law is the existence of kinematic waves which move with velocity u_c. These waves, which correspond to transverse motion of the interface, are present whenever βh is non-zero and are responsible for strong directional effects in interface correlations. The $t^{1/2}$ growth for $\theta \neq \theta^*$ is a reflection of spatial correlations in the initial state whose density profile is transported bodily through the system at speed u_c, while the subdiffusive $t^{1/3}$ growth of fluctuations along the easy axis θ^* reflects the dissipation of this profile. This picture also gives a simple way to obtain the generalization of the self-diffusion constant for the sliding tag proces, and has proved useful in a detailed study [25] of the the related problem of the non-equilibrium dynamics of a moving interface between up and down spin phases in the Toom model [26].

It is clear that a key role is played by moving density waves arising from the conservation law which, for the Ising model, is strictly valid only in the limit h/J, $T/J \to 0$. The extent to which the results would change when departures from this limit are considered is an interesting open question.

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